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LETTER TO THE EDITOR

# The two-dimensional Ising model in a magnetic field: a numerical check of Zamolodchikov's conjecture

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**Abstract.** We calculate numerically the transfer matrix spectrum of the 2D Ising model at $T = T_c$ and in a magnetic field $h \neq 0$. In the limit $h \to 0$, our results reproduce the mass spectrum recently conjectured by Zamolodchikov. Scaling functions are also studied.

The application of the principle of conformal invariance has provided a variety of new results for two-dimensional critical phenomena. Among these are the exact determination of the central charge, exponents and correlation functions for numerous physical systems.

A complete understanding of a critical point should also include the universal properties of the scaling region, where the correlation length is finite but much larger than any microscopic scale. Less progress has been made in this direction. Besides several kinds of perturbation expansions (Reinicke 1987, Saleur and Itzykson 1987, Dotsenko 1989) based on conformal invariance techniques, one of the main general results obtained so far is a quantitative version of the $c$ theorem of Zamolodchikov (1986), which allowed the calculation of several amplitude ratios (Cardy 1988a, b, Cardy and Saleur 1989).

It seems easier to consider off-critical directions which preserve integrability. The latter property is present in any conformal invariant theory, an infinite set of local integrals of motion being obtained by considering composite fields made up of $T(z)$ ($\bar{T}(\bar{z})$). As shown by Zamolodchikov (1987), some of these integrals of motion can actually survive if one perturbs the fixed-point Hamiltonian with particular relevant operators $\varphi$ to obtain a massive field theory. The cases considered so far are $\varphi_{12}$, $\varphi_{21}$ and $\varphi_{13}$ (where the indices are usual Kac labels). In such cases, the $S$ matrix factorises in terms of two particle scattering amplitudes which must satisfy the Yang–Baxter equation, bootstrap requirements as well as physical constraints. This allows, in principle, the determination of matrix elements, and in particular of the associated mass spectrum.

Using this line of thought, Zamolodchikov (1988) derived a 'minimal' $S$ matrix for the three-states Potts model perturbed by the thermal operator $\varphi_{21}$ (hence with the symmetry $Z_3$ preserved). This result was proved directly by Tsvelick (1989). There the mass spectrum corresponds simply to a particle–antiparticle pair at mass $m$, and a continuum starting at $2m$.

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In a very recent work, Zamolodchikov (1989) went on by studying the Ising model in a magnetic field, i.e. the conformal unitary model with $c=\frac{1}{2}$ perturbed by $\varphi_{12}$. Some integrals of motion can then be built explicitly which exhibit a structure related to the exceptional Lie group $E_8$. The appearance of the latter is not so surprising if one notices that $c=\frac{1}{2}$ theory, besides the usual $(A_1)_1\times(A_1)_1/(A_1)_2$, can also be obtained as the coset (Goddard *et al* 1986) $(E_8)_1\times(E_8)_1/(E_8)_2$, and suggests general connections between the coset construction and integrability. With some natural assumptions, Zamolodchikov (1989) then derived the factorised $S$ matrix. It contains eight particles with the following mass spectrum:

$$\begin{aligned}
&m_1=m \qquad m_2=2m\cos\pi/5 \qquad m_3=2m\cos\pi/30\\
&m_4=2m_2\cos 7\pi/30 \qquad m_5=2m_2\cos 2\pi/15\\
&m_6=2m_2\cos\pi/30 \qquad m_7=4m_2\cos\pi/5\cos 7\pi/30\\
&m_8=4m_2\cos\pi/5\cos 2\pi/15.
\end{aligned}\tag{1}$$

The associated scattering theory is not yet fully understood; in particular the masses $m_4\ldots m_8$ are above the cut starting at $2m$, and the reason for their stability is unknown.

In this letter, we would like to report numerical calculations which confirm the spectrum (1).

For technical reasons, it is more convenient to consider a very anisotropic limit where the Ising model transfer matrix reduces to the exponential of the following quantum Hamiltonian (Kogut 1979):

$$H=-\sum_{n=1}^{N}\left(t\sigma^x(n)+\sigma^z(n)\sigma^z(n+1)+h\sigma^z(n)\right)\tag{2}$$

where the $\sigma$ are Pauli matrices. $t$ is related to the temperature and, for $T=T_c$, $t=1$, to which we restrict ourselves here. $h$ is similarly related to the usual magnetic field, and $h=0$ at criticality. The masses $m_i$ are simply obtained as $m_i=E_i-E_0$ where $E_0$ is the ground-state energy. We are interested in the scaling region, when $h\to 0$, $N\to\infty$ such that the product

$$\mu=hN^{15/8}\tag{3}$$

remains finite. In this limit, gaps are expected to satisfy a scaling law

$$m_i=h^{8/15}G_i(\mu).\tag{4}$$

This essentially does not depend on the anisotropic limit.

Since $H$ is invariant under translations (we take periodic boundary conditions), its eigenvalues can be labelled by the momenta. The translation operator $T$ is

$$T=\exp(\mathrm{i}P)=\exp\left(\frac{2\pi\mathrm{i}}{N}k\right).\tag{5}$$

On a finite lattice, $k$ takes the values $0,1,\ldots,N-1(\mathrm{mod}\ N)$ but, in contrast to the situation found precisely *at* the critical point, it is $P$ and not $k$ which should be interpreted as momentum, when taking the continuum limit.

What was known hitherto about the functions $G_i(\mu)$ is the limit $\mu\to 0$, which is related in the standard way to critical dimensions (for instance $\lim_{\mu\to 0}(m_2/m_1)=8$). The prediction of Zamolodchikov (1989) concerns the other extreme case $\mu\to\infty$.

For several values of $h$, $H$ is diagonalised by standard methods. The extrapolation for the limit $N\to\infty$ is done using the BST extrapolation algorithm (Bulirsch and Stoer

1964, Henkel and Schütz 1988). Qualitatively, the ratios $r_i = m_{i+1}/m_i$ behave as follows. For very small $N$, where $N$ is small compared with the correlation length $\xi$, the $r_i$ are close to the values predicted by conformal invariance (e.g. $r_1 \simeq 8$, see Henkel (1987) and references therein). As $N$ increases $r_i$ decreases until it reaches a minimum (e.g. $r_1 \simeq 1.35$, at $\mu \simeq 8$), and only after this does $r_i$ tend to its $N \to \infty$ limit (where $\xi \ll N$). The existence of this minimum makes the extrapolation for $N \to \infty$ very difficult. For small values of $h$, the minimum occurs for values of $N$ which are far too large to allow for a numerical diagonalisation of $H$. Consequently, we have to consider relatively large values of $h$. In table 1, we give some estimates for the ratios $r_1$ and $r_2$.

Since Zamolodchikov's (1989) result (1) can only be applied to our model (2) in the limit $h \to 0$, the data of table 1 should be extrapolated back to $h \to 0$ and then compared with 1.618 and 1.989, respectively. We find that our data are consistent with the theoretical prediction. Similarly, higher ratios are compatible with a continuum starting at 2.

Let us briefly discuss the case $k \neq 0$. First, the spectrum does not seem to depend on $k$, as illustrated in table 2. This comes from the fact that choosing a non-zero $k$ does not change the exponential decay of, say, the spin-spin correlation function but merely projects out some angle-dependent finite-size correction term. On the other hand, we can check the energy-momentum relation of a massive free particle in a Brillouin zone (see equation (5))

$$E^2 = m^2 + 4\sin^2\frac{P}{2}. \tag{6}$$

This is done in figure 1 for $m_1$ and $m_2$, where a straight line is obtained. Small deviations are likely due to finite-size (field) effects and to the fact that particles are expected to become free only when $h \to 0$. We note, however, that $E$ should be rescaled by some suitable factor $\gamma$ in order to reproduce (6). This observation can be used to fix the ($h$-dependent) overall normalisation of $H$.

**Table 1.** Estimates for the ratios $r_i = m_{i+1}/m_1$ of the inverse correlation lengths a function of the magnetic field, as obtained from the BST extrapolation algorithm. The numbers in brackets give the error in the last digit.

| $h$ | $r_1$ | $r_2$ |
|---|---|---|
| 0.2 | 1.610 (5) | 1.98 (2) |
| 0.3 | 1.604 9 (3) | 1.968 (8) |
| 0.5 | 1.594 68 (3) | 1.948 (1) |
| 0.8 | 1.580 710 (1) | 1.9242 (1) |

**Table 2.** Estimates for the low-lying spectrum, for $t = 1.0$ and $h = 0.3$.

| $k$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ |
|---|---|---|---|---|---|
| 0 | 5.385 31 | 8.643 1 | 10.58 | 11.2 | 12 |
| 1 | 5.385 9 | 8.685 | 10.64 | 12.7 | — |
| 2 | 5.385 | 8.653 | 10.7 | 11 | — |
| 3 | 5.39 | 8.65 | 10.7 | — | — |
| 4 | 5.37 | 8.6 | — | — | — |

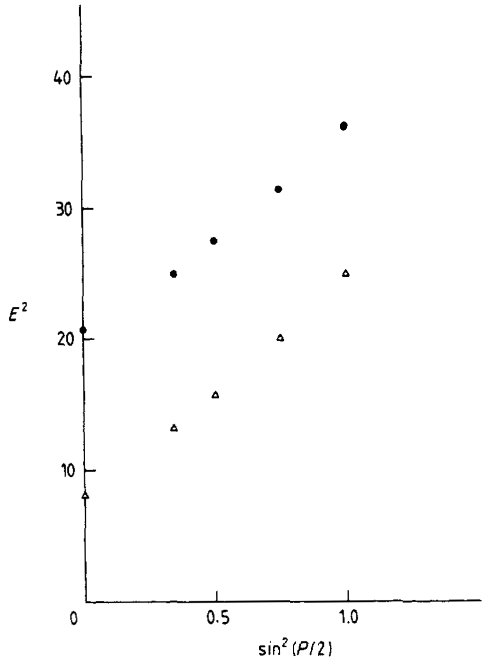


**Figure 1.** Check of the energy-momentum relation (6) for $h = 0.3$ ($\triangle$: $m_1$, $\bullet$: $m_2$).

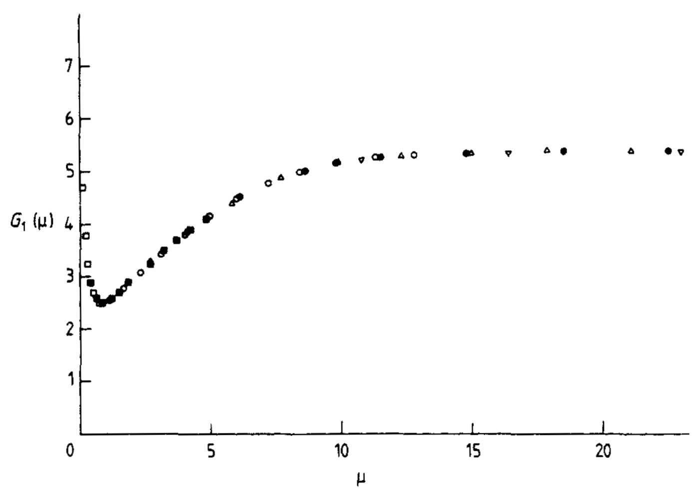


**Figure 2.** Scaling function $G_1(\mu)$ for the lowest gap of $H$, corresponding to the spin-spin correlation length. The symbols refer to different values of $h$ ($\square$: 0.01, $\blacksquare$: 0.03, $\bigcirc$: 0.08, $\triangle$: 0.2, $\bullet$: 0.3, $\nabla$: 0.8).

While (1) applies to the limit $\mu \to \infty$, we now consider the whole scaling functions. In figures 2 and 3 we give $G_1(\mu)$ and $G_2(\mu)$, which do satisfy the scaling law (3). We note that for $\mu \to 0$, the scaling function $G_i(\mu)$ shows a power-law singularity. The latter is nothing else than the familiar term known from conformal invariance. Writing (note the normalisation of $H$ (2))

$$G_i(\mu) = 4\pi x_i \mu^{-8/15} + H_i(\mu) \tag{7}$$

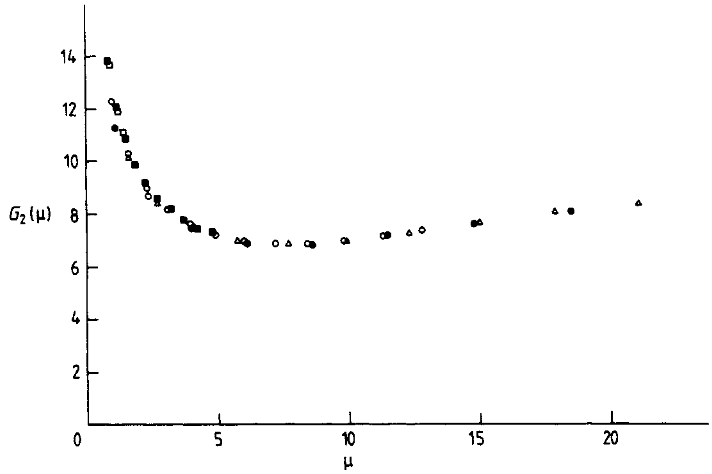


**Figure 3.** Scaling function $G_2(\mu)$ for the second gap, corresponding to the energy-energy correlation length. The symbols are the same as in figure 1.

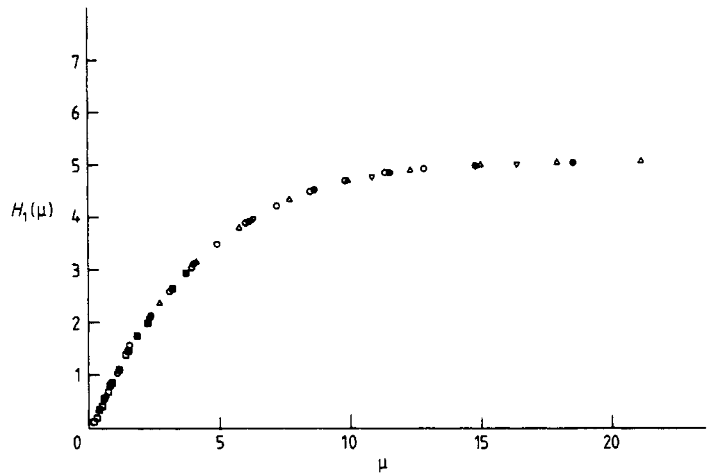


**Figure 4.** Reduced scaling function $H_1(\mu)$.

where $x_i$ is the critical exponent obtained at $h=0$, the singularity is substracted. We give the reduced scaling functions $H_1(\mu)$ and $H_2(\mu)$ in figures 4 and 5. We note that for small values of $\mu$, where we considered values of $\mu$ down to $\mu \simeq 0.2$, $H_1(\mu)$ seems to be a linear function of $\mu$ while $H_2(\mu)$ appears to depend logarithmically on $\mu$. For comparison, we list the scaling functions for the case of a thermal perturbation ($z = N|t-1|$, $h=0$) (Henkel 1987)

$$\begin{aligned} m_1 &= |t-1|\left(\frac{\pi}{2z}+1+\frac{\ln 2}{\pi}z+\mathrm{O}(z^3)\right) \\ m_2 &= |t-1|\left(\frac{4\pi}{z}+\frac{2}{\pi}z+\mathrm{O}(z^3)\right). \end{aligned} \tag{8}$$

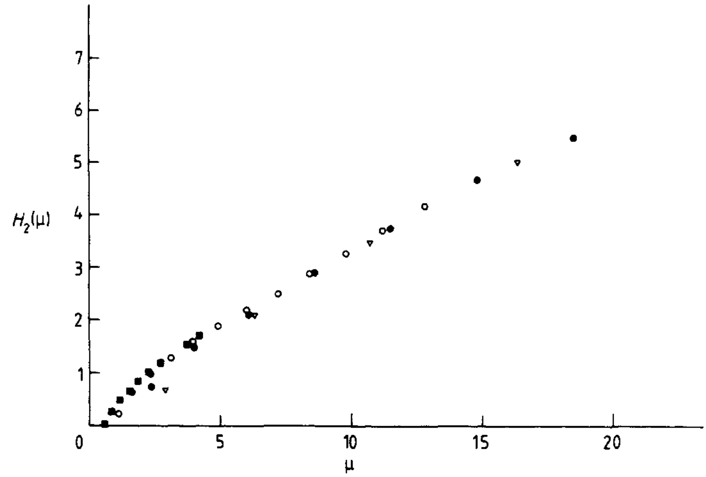


**Figure 5.** Reduced scaling function $H_2(\mu)$. Note that scaling is relatively bad for small values of $\mu$.

In general, conformal invariance and scaling theory provide the powers of $\mu$ (or $z$) which should enter into the expansion of the $H_i(\mu)$. Probably the data of figures 4 and 5 are still affected by finite-size corrections so that the scaling regime is not yet reached completely.

To summarise. We have studied the masses of the 2D Ising model in a magnetic field, trying to confirm the prediction (1) concerning the $\mu \to \infty$ limit. Since the $G_i(\mu)$ have the form (7) with $H_i(\mu)$ slowly growing with $\mu$, the limit is hard to obtain numerically. However, our data (see table 1) do support Zamolodchikov's prediction.

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